Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT

Journal of Approximation Theory 124 (2003) 7–24

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Universality and chaos for tensor products of operators

Félix Martínez-Giménez[a] and Alfredo Peris[b,*,1]

[a] *E.T.S.I. Agrónomos, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E-46071 Valencia, Spain*

[b] *E.T.S. Arquitectura, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E-46071 Valencia, Spain*

Received 24 September 2001; accepted in revised form 2 June 2003

Dedicated to Professor Manuel Valdivia on the occasion of his seventieth birthday

Communicated by Peter B. Borwein

Abstract

We give sufficient conditions for the universality of tensor products $\{T_n \widetilde{\otimes} R_n : n \in \mathbb{N}\}$ of sequences of operators defined on Fréchet spaces. In particular we study when the tensor product $T \widetilde{\otimes} R$ of two operators is chaotic in the sense of Devaney. Applications are given for natural operators on function spaces of several variables, in Infinite Holomorphy, and for multiplication operators on the algebra $L(E)$ following the study of Kit Chan.

Keywords: Universality; Hypercyclic vectors; Tensor products of operators; Chaotic dynamics

A sequence $\{T_n : E \to E : n \in \mathbb{N}\}$ of operators on a Fréchet space E is called *universal* provided there exists some universal vector $x \in E$ such that

$$\overline{\{T_n x : n \in \mathbb{N}\}} = E.$$

In other words: Every element in E can be approximated by elements in the orbit of x. An operator T on E is said to be *hypercyclic* if the sequence of its iterates

*Corresponding author. Fax: 34-96-387-7669.

E-mail addresses: fmarting@mat.upv.es (F. Martínez-Giménez), aperis@mat.upv.es (A. Peris).

[1] Supported by MCYT, Proy. BFM2001-2670.

0021-9045/03/$ - see front matter

doi:10.1016/S0021-9045(03)00118-7

$\{T^n : n \in \mathbb{N}\}$ is universal, i.e., T is hypercyclic if there exists some $x \in E$ with dense orbit

$$\overline{\mathrm{Orb}(T,x)} := \overline{\{T^n x : n \in \mathbb{N}\}} = E.$$

If in addition T admits a dense subset $\mathscr{P} \subset E$ of periodic points, then T is *chaotic* in the sense of Devaney [15] (see e.g. [3]).

Our main purpose in this paper is to give a general tool which permits us to obtain hypercyclicity, universality or chaos for operators that can be represented as tensor products of simpler operators. This is the case, e.g., of many operators defined on function spaces of several variables (translations, partial differential operators, etc.) that admit a tensorial representation with factors consisting of operators on function spaces of one variable.

The study of the chaotic behavior of linear operators on infinite dimensional spaces goes back to Birkhoff in 1929 [7]. He showed that the translation operator $T_a : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$, $f(z) \mapsto f(z+a)$, $(a \neq 0)$, is hypercyclic on the Fréchet space $\mathscr{H}(\mathbb{C})$ of entire functions. This result was generalized by Godefroy and Shapiro [19] who proved that every operator on $\mathscr{H}(\mathbb{C}^N)$ that commutes with all translations, and is not a scalar multiple of the identity, is chaotic. As a consequence they also obtained that every partial differential operator on $\mathbb{R}^N$, which is not a scalar multiple of the identity, is chaotic on the Fréchet space $C^\infty(\mathbb{R}^N)$. In [8] hypercyclicity of convolution operators on some Fréchet spaces of C^∞-functions is studied. The existence of hypercyclic operators on arbitrary separable Fréchet spaces was proved in [11] extending the Banach space result of Ansari and Bernal. However, not every separable Banach space admits a chaotic operator [9]. During the last years there have been many contributions to the topic by several authors, especially in the Banach space setting. The article of Grosse–Erdmann [20] constitutes an exhaustive up to 1999 survey of universal sequences of operators and hypercyclicity. Supercyclic operators yield an interesting type of universality. We recall T is *supercyclic* if there is a vector in the space whose multiples of the orbit form a dense set. Equivalently, if the sequence $\{\lambda_n T^n : n \in \mathbb{N}\}$ is universal, where (λ_n) is some sequence in the scalar field. Although not specifically stated, many of the results in this note remain valid for supercyclicity.

In contrast with non-linear phenomena, where a rigorous establishment of chaotic behavior is usually a difficult task, there are "computable" sufficient conditions for hypercyclicity (and universality) of linear operators. These conditions are essentially summarized in the so-called Hypercyclicity Criterion. It is not known whether every hypercyclic operator satisfies this criterion, but many classes of hypercyclic operators (e.g., chaotic operators [6]) fulfill the Hypercyclicity Criterion.

In the first section of this paper we recall basic definitions and facts about universality and tensor products and we give all the general results concerning universality, hypercyclicity and chaos of tensor products of operators. It is not surprising that the Universality Criterion is involved in sufficient conditions for universality of tensor products. However, this criterion is just needed in one of the factors of the tensor product, while much weaker conditions are required in the second factor. In particular we obtain that the tensor product $T \widetilde{\otimes} I$ of an operator T

satisfying the Hypercyclicity Criterion, with the identity operator I, is hypercyclic. We finish this section with the comparison of stability properties of hypercyclicity by taking direct sums or tensor products. It is shown that $T \oplus T$ is hypercyclic if and only if $T \widetilde{\otimes} I$ is hypercyclic, which gives another equivalent formulation (in the context of tensor products) of an old problem of Herrero.

The second section is devoted to examples and applications of the techniques introduced in the first section. We characterize hypercyclicity of tensor products of unilateral weighted backward shifts and we give an easier alternative proof of a result of Abe and Zappa concerning universality of translations in $\mathscr{H}(\mathbb{C}^N)$. The methods of Section 1 also allow us to study the universality of multiplication operators on the algebra $L(E)$ endowed with the strong operator topology (SOT), for separable Banach spaces E. This line of work was initiated by Kit Chan [12] who showed that left multiplication operators $L_T : B(H) \to B(H)$, $R \mapsto TR$, on the algebra $B(H)$ of bounded operators on a separable Hilbert space with the SOT are hypercyclic provided that T satisfies the Hypercyclicity Criterion. We generalize his results using a different approach. We close the section with results concerning hypercyclicity and chaos of certain composition operators in Infinite Holomorphy. The results of this paper are part of F. Martínez-Giménez's Ph.D. thesis [23], under the advice of A. Peris.

1. Tensor stability of universality and chaos

From now on $\{T_n : E \to E : n \in \mathbb{N}\}$ will denote a sequence of (continuous and linear) operators on a separable Fréchet space E. We will give a general sufficient condition for universality. This condition is inspired in the so-called Hypercyclicity Criterion given by Kitai [22] in her unpublished Ph.D. thesis and rediscovered by Gethner and Shapiro [18]. We use the general form of this Criterion as given in [6]. The proof of universality for operators satisfying the Criterion is based on a Baire category argument (see [20]). Even more, under the hypothesis of the following definition, there exists a dense G_δ set of universal vectors. Hypercyclicity is a particular case.

Definition 1.1. We say that $\{T_n : n \in \mathbb{N}\}$ satisfies the Universality Criterion (UC) provided there exist X and Y dense subsets of E and maps $S_n : Y \to E$, $n \in \mathbb{N}$, such that

(i) $T_n x \xrightarrow{n \to \infty} 0$ for all $x \in X$,

(ii) $S_n y \xrightarrow{n \to \infty} 0$ for all $y \in Y$,

(iii) $(T_n \circ S_n) y \xrightarrow{n \to \infty} y$ for all $y \in Y$.

An operator T on E is said to satisfy the Hypercyclicity Criterion (HC) with respect to an increasing sequence of positive integers (n_k), if the sequence of iterates $\{T^{n_k} : k \in \mathbb{N}\}$ satisfies the Universality Criterion.

Operators T satisfying the HC are characterized in [6] as those such that $T \oplus T$ is hypercyclic. This result established an equivalence between an open problem of Herrero [21], which asks if $T \oplus T$ is hypercyclic whenever T is, and the interesting question of whether every hypercyclic operator on a separable Fréchet space satisfies the HC.

Remark 1.2. (1) Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of operators and (n_k) an increasing sequence of positive integers. It is easy to see that if $\{T_n : n \in \mathbb{N}\}$ satisfies the UC then $\{T_{n_k} : k \in \mathbb{N}\}$ also satisfies the UC, and that the existence of universal vectors for $\{T_{n_k} : k \in \mathbb{N}\}$ implies the universality of $\{T_n : n \in \mathbb{N}\}$. This observation, together with the convenience of presentation, motivates us to define the UC for the whole sequence of integers instead of stating it in terms of subsequences.

(2) Without loss of generality we can suppose (and we will) that the dense subsets $X, Y \subset E$ are actually subspaces and that the maps S_n are linear in the above criterion. This can be done by considering algebraic basis, contained in X and Y respectively, of the span of these sets, and then linearly extend the original S_n to the span of Y.

Shapiro [27] gave another useful condition, known as the *Hypercyclicity Comparison Principle* (HCP): If $T : (E, \tau) \to (E, \tau)$ is such that there is a dense subspace $F \hookrightarrow E$ and a finer topology τ' on F such that $T|_F : (F, \tau') \to (F, \tau')$ is hypercyclic, then T is hypercyclic. In [24] we generalized this result via commutative diagrams. The following lemma is a consequence and we will use it repeatedly through the paper.

Lemma 1.3. *Let $T_{i,n} : E_i \to E_i : n \in \mathbb{N}$ be a sequence of operators on a separable Fréchet space E_i, $i = 1, 2$, and let $\phi : E_1 \to E_2$ be a continuous map with dense range such that $T_{2,n} \circ \phi = \phi \circ T_{1,n}$, for all $n \in \mathbb{N}$. That is, the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{T_{1,n}} & E_1 \\ \phi \downarrow & & \phi \downarrow \\ E_2 & \xrightarrow{T_{2,n}} & E_2 \end{array}$$

commutes for all $n \in \mathbb{N}$. If $\{T_{1,n} : n \in \mathbb{N}\}$ is a universal sequence (satisfies the UC), then $\{T_{2,n} : n \in \mathbb{N}\}$ is also a universal sequence (satisfies the UC). For single operators $T_i : E_i \to E_i$, $i = 1, 2$, such that $T_2 \circ \phi = \phi \circ T_1$ we have:

(1) *If T_1 is hypercyclic, then T_2 is also hypercyclic.*
(2) *If T_1 is chaotic, then T_2 is also chaotic.*
(3) *If T_1 satisfies the HC, then T_2 also satisfies the HC.*

The goal of this section is to study conditions under which tensor products of operators are universal, hypercyclic or chaotic. A reasonable guess is that tensorizing sequences of operators that satisfy the UC should lead to a universal sequence.

However, we can give a stronger result since conditions on one of the factors can be actually weakened. We have to introduce a new concept that is related to the UC.

Definition 1.4. We say that a sequence $\{T_n : n \in \mathbb{N}\}$ of operators on E satisfies the Tensor Universality Criterion (TUC) if there exist dense subsets X and Y of E, and maps $S_n : Y \to E,\ n \in \mathbb{N}$, such that

(i) $\{T_n x\}_{n=1}^{\infty}$ is bounded for each $x \in X$,
(ii) $\{S_n y\}_{n=1}^{\infty}$ is bounded for each $y \in Y$,
(iii) $(T_n \circ S_n)y \xrightarrow{n\to\infty} y$ for all $y \in Y$.

An operator T on E satisfies the Tensor Hypercyclicity Criterion (THC) with respect to an increasing sequence of positive integers (n_k), provided the sequence of iterates $\{T^{n_k} : k \in \mathbb{N}\}$ satisfies the TUC.

Remark 1.5. In a similar way to Remark 1.2, it is easy to see that if $\{T_n : n \in \mathbb{N}\}$ satisfies the TUC and (n_k) is an increasing sequence of positive integers, then $\{T_{n_k} : k \in \mathbb{N}\}$ also satisfies the TUC. Also, without loss of generality, we may (and will) suppose that the subsets X and Y in the previous definition are in fact subspaces of E and that the maps S_n are linear.

Example 1.6. (1) Obviously, sequences of operators satisfying the UC satisfy the TUC.

(2) Any (surjective) isometry on a Banach space satisfies the THC with respect to the sequence of all positive integers.

(3) If an operator R on a Fréchet space E has a dense set of periodic points, then R satisfies the THC with respect to the sequence of positive integers. Indeed, let X be the set of periodic points of R. Then $\{R^k x\}_{k\geqslant 1}$ is bounded for each $x \in X$, since the orbit of a periodic point is finite, and $R_{|X} : X \to X$ is a bijection. We define $S_k := S^k,\ k \in \mathbb{N}$, where $S := (R_{|X})^{-1}$. In particular, if $R^n = I$ (the identity operator) for some $n \in \mathbb{N}$, then R satisfies the THC.

Let us establish the necessary notation and preliminaries on tensor products of locally convex spaces. For a complete description we refer the reader to the book of Defant and Floret [14, Section 35]. We will first recall the definition of the projective tensor norm, associated with the nuclear norm of operators. If G is a locally convex space then $\mathrm{cs}(G)$ denotes the set of all continuous seminorms on G.

Definition 1.7. Given G and H locally convex spaces, the projective topology π on the tensor product $G \otimes H$ is defined as the locally convex topology generated by the family of seminorms $\mathscr{A}_\pi := \{p \otimes_\pi q : p \in \mathrm{cs}(G),\ q \in \mathrm{cs}(H)\}$, where

$$p \otimes_\pi q(z) := \inf\left\{ \sum_{j=1}^{n} p(x_j)q(y_j) : z = \sum_{j=1}^{n} x_j \otimes y_j \right\}, \quad z \in G \otimes H.$$

For elementary tensors $z = x \otimes y$ we just have $p \otimes_\pi q(z) = p(x)q(y)$. With this topology the space is denoted by $G \otimes_\pi H$, and its completion is $G \widetilde{\otimes}_\pi H$.

Given a tensor norm "a" (see [14, Chapter II]), we have the corresponding locally convex topology on the tensor product $G \otimes H$ of two locally convex spaces G and H. The space is then denoted by $G \otimes_a H$. The metric mapping property for tensor norms yields that the operator $T_1 \otimes T_2 : G_1 \otimes_a G_2 \to H_1 \otimes_a H_2$ is continuous whenever $T_1 : G_1 \to H_1$ and $T_2 : G_2 \to H_2$ are continuous operators between locally convex spaces (see [14, 35.2]). On the other hand the projective topology is finer than the a-topology on $G \otimes H$. These two properties constitute all we need to know about tensor norms for our purposes.

We can now state the main result of this section.

Theorem 1.8. *Let E and F be separable Fréchet spaces. If the sequence of operators $\{T_n^1 : E \to E : n \in \mathbb{N}\}$ satisfies the UC, and the sequence of operators $\{T_n^2 : F \to F : n \in \mathbb{N}\}$ satisfies the TUC, then*

$$\{T_n^1 \widetilde{\otimes} T_n^2 : E \widetilde{\otimes}_a F \to E \widetilde{\otimes}_a F : n \in \mathbb{N}\}$$

satisfies the UC, and therefore is universal, for any tensor norm a.

Proof. Let $X^1, Y^1 \subset E$, $X^2, Y^2 \subset F$ be dense subspaces and $S_n^1 : Y^1 \to E$, $S_n^2 : Y^2 \to F$, $n \in \mathbb{N}$, linear maps satisfying the conditions of UC and TUC for the sequences of operators $\{T_n^1 : E \to E : n \in \mathbb{N}\}$ and $\{T_n^2 : F \to F : n \in \mathbb{N}\}$, respectively. We will see that $X := X^1 \otimes X^2$, $Y := Y^1 \otimes Y^2$ and the maps

$$S_n := S_n^1 \otimes S_n^2 : Y \to E \otimes F, \quad n \in \mathbb{N},$$

are such that conditions (i)–(iii) of the UC are satisfied for the sequence of operators

$$\{T_n := T_n^1 \widetilde{\otimes} T_n^2 : E \widetilde{\otimes}_a F \to E \widetilde{\otimes}_a F : n \in \mathbb{N}\}.$$

The fact that the maps S_n^i, $i = 1, 2$, $n \in \mathbb{N}$, are linear is essential in order to have that $S_n(z)$ is independent of the representation of z as a tensor and, therefore, to get that the maps S_n are well defined.

It will suffice to show that convergences in (i)–(iii) are valid for the π-topology (which is the finest one). To do this, let $p \in \mathrm{cs}(E)$ and $q \in \mathrm{cs}(F)$. Then, if we compute on elementary tensors, we have

$$\lim_n p \otimes_\pi q((T_n^1 \otimes T_n^2)(x_1 \otimes x_2)) \\ = \lim_n p(T_n^1 x_1) q(T_n^2 x_2) = 0, \quad \forall x_i \in X_i, \ i = 1, 2,$$

since the first sequence tends to 0 and the second one is bounded. Analogously

$$\lim_n p \otimes_\pi q(S_n(y_1 \otimes y_2)) = 0, \quad \forall y_i \in Y_i, \ i = 1, 2.$$

This implies conditions (i) and (ii) of the UC by taking linear combinations of elementary tensors. Finally,

$$\begin{aligned}&\lim_n p\otimes_\pi q(T_nS_n(y_1\otimes y_2)-(y_1\otimes y_2))\\&\quad=\lim_n p\otimes_\pi q(T_nS_n(y_1\otimes y_2)-(y_1\otimes T_n^2S_n^2y_2)+(y_1\otimes T_n^2S_n^2y_2)-(y_1\otimes y_2))\\&\quad\leqslant\lim_n[p(T_n^1S_n^1(y_1)-y_1)q(T_n^2S_n^2(y_2))\\&\qquad+p(y_1)q(T_n^2S_n^2(y_2)-y_2)]=0,\quad\forall y_i\in Y_i,\ i=1,2,\end{aligned}$$

which completes the proof. □

Remark 1.9. Universality is a phenomenon that occasionally occurs for operators $T_n: G\to H$ with different domain and range spaces. Corresponding UC and TUC criteria can be defined, and a result which is analogous to theorem above holds if we assume G Fréchet, and H metrizable and separable. We did not follow this framework for convenience of the presentation.

The corresponding version for hypercyclicity is now a consequence of this theorem.

Corollary 1.10. *Let E and F be separable Fréchet spaces, and $T: E\to E$, $R: F\to F$, operators such that T satisfies the HC and R satisfies the THC (with respect to a common sequence of integers), then*

$$T\widetilde{\otimes}R: E\widetilde{\otimes}_a F\to E\widetilde{\otimes}_a F$$

satisfies the HC (and therefore is hypercyclic) for any tensor norm a.

An interesting particular case can be derived from Examples 1.6 (3).

Corollary 1.11. *Let E and F be separable Fréchet spaces and $T: E\to E$ an operator satisfying the HC. If the operator $R: F\to F$ has a dense set of periodic points, then*

$$T\widetilde{\otimes}R: E\widetilde{\otimes}_a F\to E\widetilde{\otimes}_a F$$

is hypercyclic for any tensor norm a.

This corollary and the fact that chaotic operators satisfy the HC [6, Proposition 2.14], have an important consequence about chaos of tensor products of operators.

Corollary 1.12. *If E and F are separable Fréchet spaces, the operator $T_1: E\to E$ is chaotic, and $T_2: F\to F$ has a dense set of periodic points, then*

$$T_1\widetilde{\otimes}T_2: E\widetilde{\otimes}_a F\to E\widetilde{\otimes}_a F$$

is chaotic for any tensor norm a. In particular $T_1\widetilde{\otimes}T_2$ is chaotic if T_1 and T_2 are so.

Proof. By [6, Proposition 2.14], T_1 satisfies the HC. By Corollary 1.11 we have that $T_1 \tilde{\otimes} T_2$ is hypercyclic on $E \tilde{\otimes}_a F$.

Let $\mathscr{P}(T_1) \subset E$ and $\mathscr{P}(T_2) \subset F$ be the sets of periodic points for T_1 and T_2, respectively. We first observe that $\mathscr{P}(T_1)$ and $\mathscr{P}(T_2)$ are dense linear spaces, hence $\mathscr{P}(T_1) \otimes \mathscr{P}(T_2)$ is a dense subspace of $E \tilde{\otimes}_a F$. On the other hand each point of $\mathscr{P}(T_1) \otimes \mathscr{P}(T_2)$ is a periodic point for $T_1 \otimes T_2$. □

It is interesting to observe that chaotic operators T_1 and T_2 which do not satisfy the HC with respect to a common sequence of integers can be constructed, but $T_1 \tilde{\otimes} T_2$ should be chaotic in view of the previous result. This can be done, for instance, with weighted shifts.

Remark 1.13. Those readers interested in transitivity or chaos for non-metrizable or non-separable spaces might wonder about these properties for tensor products of operators defined on arbitrary locally convex spaces. We notice that the HC always implies transitivity, and that neither separability nor metrizability was necessary in the proof of Theorem 1.8 to obtain the Criterion in the tensor product. Therefore, if T and R are defined on general locally convex spaces, T satisfies the HC, and R satisfies the THC, then $T \otimes R$ is transitive.

We would like to finish this section with a comparison of hypercyclicity of direct sums with hypercyclicity of tensor products. It is known that if T and R satisfy the HC, with respect to a common sequence of integers, so does the direct sum $T \oplus R$. The same is true for tensor products as a consequence of Corollary 1.10. However, if $T \oplus R$ is hypercyclic then T and R are also hypercyclic, while it may happen that $T \tilde{\otimes} R$ is hypercyclic with neither T nor R hypercyclic. If we take $T := 2I$ and the backward shift $R := B$, as operators defined on l_2, then it is evident that none of them is hypercyclic. But $T \tilde{\otimes} R = I \tilde{\otimes} 2B$, with the operator $2B$ satisfying the HC and I satisfying the THC (both with respect to the whole sequence of integers). Corollary 1.10 gives the hypercyclicity of $T \tilde{\otimes} R$. The following result clarifies the connection between hypercyclicity of tensor products and hypercyclicity of direct sums, and yields another equivalent formulation (in the context of tensor products) of an old problem of Herrero [21].

Proposition 1.14. *Let E be a separable Fréchet space, $T : E \to E$ an operator, and F a separable Fréchet space with* $\dim(F) \geqslant 2$. *The following are equivalent*:

(1) *T satisfies the HC*,
(2) *$T \tilde{\otimes} I : E \tilde{\otimes}_a F \to E \tilde{\otimes}_a F$ is hypercyclic for any* (*some*) *tensor norm a.*
(3) *$T \oplus T : E \oplus E \to E \oplus E$ is hypercyclic.*

Proof. (1) → (2) This is a consequence of Corollary 1.11 by taking $R = I$.

(3) → (1) This is one of the implications in [6, Theorem 2.3].

(2)→(3) Since $\dim(F) \geqslant 2$ there exist linearly independent $f_1^*, f_2^* \in F'$. Define then $\phi : E\widetilde{\otimes}_a F \to E \oplus E$,

$$\phi\Big(\sum e_i \otimes f_i\Big) := \Big(\sum e_i \langle f_i, f_1^* \rangle, \ \sum e_i \langle f_i, f_2^* \rangle\Big).$$

The operator ϕ is surjective (given $e_1, e_2 \in E$, we take $f_1, f_2 \in F$ such that $\langle f_i, f_j^* \rangle = \delta_{ij}$ and we have $\phi(e_1 \otimes f_1 + e_2 \otimes f_2) = (e_1, e_2)$). On the other hand

$$\begin{aligned}(T \oplus T)\Big(\phi\Big(\sum e_i \otimes f_i\Big)\Big) &= \Big(\sum Te_i \langle f_i, f_1^* \rangle, \ \sum Te_i \langle f_i, f_2^* \rangle\Big) \\ &= \phi\Big((T\widetilde{\otimes} I)\Big(\sum e_i \otimes f_i\Big)\Big),\end{aligned}$$

that is, the diagram

$$\begin{array}{ccc} E\widetilde{\otimes}_a F & \xrightarrow{T\widetilde{\otimes} I} & E\widetilde{\otimes}_a F \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ E \oplus E & \xrightarrow{T \oplus T} & E \oplus E \end{array}$$

is commutative. Lemma 1.3 completes the proof. □

We are ready to formulate an equivalent problem to Herrero's, which asks whether $T \oplus T$ is hypercyclic whenever T is so. As a consequence of [6, Theorem 2.3], our problem is also equivalent to the question about the existence of hypercyclic operators not satisfying the HC.

Problem 1.15. *Is $T\widetilde{\otimes} I : E\widetilde{\otimes}_\pi E \to E\widetilde{\otimes}_\pi E$ hypercyclic for every hypercyclic operator T on a separable Fréchet space E?*

The case of the tensor product of an operator with itself seems to be even more intriguing.

Problem 1.16. *Let T be an operator on a separable Fréchet space. If T is hypercyclic, is $T\widetilde{\otimes} T$ hypercyclic? Is the converse true?*

2. Examples and applications

This section is devoted to applications of the general theory described in the previous section. We show a few possible future trends of applications of tensor product techniques in the context of universality, chaos, and cyclic behavior of operators. However, it is not our purpose to cover exhaustively many examples. The first example shows how to obtain universality of operators on function spaces of several variables, based on the corresponding result for one variable. It was our aim to characterize hypercyclicity of tensor products of operators within a "nice class" of operators. We do so for the class of weighted backward shift operators. It also turns out that the study of universality of multiplication operators on the algebra $L(E)$, as a continuation of the work initiated by Chan [12], fits within the framework of tensor products in a natural way: Proofs are much simpler and the results obtained are stronger. Finally we study hypercyclicity and chaos of certain composition operators in Infinite Holomorphy via tensor products.

2.1. Operators on function spaces of several variables

In 1929 Birkhoff [7] proved the universality of translation operators on $\mathscr{H}(\mathbb{C})$ endowed with the topology of uniform convergence on compact sets. Godefroy and Shapiro [19] showed that every operator on $\mathscr{H}(\mathbb{C}^N)$ that commutes with all translations, and is not a scalar multiple of the identity, is chaotic. A generalization to $\mathscr{H}(\mathbb{C}^N)$ of Birkhoff's universality is due to Abe and Zappa [1, Section 2] (see below). A much stronger result is presented in [4, Theorem 8]. Cyclicity (another version of universality) of translation operators defined on certain function spaces of several variables was studied in [25, Section 3].

Abe–Zappa's result: *Let $\mathscr{H}(\mathbb{C}^N)$ be the space of holomorphic functions on $\mathbb{C}^N$ endowed with the topology of uniform convergence on compact sets. For $a\in\mathbb{C}^N$, T_a denotes the translation operator*

$$T_a : \mathscr{H}(\mathbb{C}^N)\to\mathscr{H}(\mathbb{C}^N) : T_af(z) := f(z+a).$$

Let $\{a^{(j)}\}_{j\geqslant 1}$ be a sequence in $\mathbb{C}^N$ with $\sup_j ||a^{(j)}|| = \infty$. Then the sequence $\{T_{a^{(j)}}\}_{j\geqslant 1}$ is universal.

Here is a proof using our tensor product techniques: We recall that $\mathscr{H}(\mathbb{C}^N)$ can be represented as the tensor product $\widetilde{\otimes}_{N,\pi}\mathscr{H}(\mathbb{C})$ (it is enough to consider the extension of the operator $f_1(z_1)\otimes\cdots\otimes f_N(z_N)\mapsto f(z_1,\dots,z_N) := f_1(z_1)\cdots f_N(z_N)$), and that the operator T_a on $\mathscr{H}(\mathbb{C}^N)$ coincides with $\widetilde{\otimes}^N_{\pi\ i=1} T_{a_i}$ on $\widetilde{\otimes}_{N,\pi}\mathscr{H}(\mathbb{C})$. Since $\sup_j ||a^{(j)}|| = \infty$, there exists an index $1\leqslant i_0\leqslant N$ such that $\sup_j |a^{(j)}_{i_0}| = \infty$. Without loss of generality we will assume that $i_0 = 1$, that $\lim_{j\to\infty} |a^{(j)}_1| = \infty$ and that there exists $\alpha := \lim_{j\to\infty}\frac{a^{(j)}_1}{|a^{(j)}_1|}$ (if not we pass to a suitable subsequence).

In this situation it is easy to prove that $\{T_{a^{(j)}_1}\}_{j\geqslant 1}$ satisfies the UC: We follow Godefroy–Shapiro's approach and define

$$X := \operatorname{span}\{e^{\lambda z} : |e^{\alpha\lambda}| < 1\},\quad Y := \operatorname{span}\{e^{\lambda z} : |e^{\alpha\lambda}| > 1\}.$$

A Hahn–Banach argument shows that X and Y are dense in $\mathscr{H}(\mathbb{C})$ (see e.g. [19, Section 5]). Alternatively it would be possible to give a direct argument. Let K be a compact set of $\mathbb{C}$ and $R > 0$ such that $|z|\leqslant R$ for each $z\in K$. For any function

$f(z) = e^{\lambda z}$ of X and for each z of K we have

$$|(T_{a_1^{(j)}} f)(z)| \leqslant e^{|\lambda|R} |e^{\lambda a_1^{(j)}}| = e^{|\lambda|R} \left| e^{\lambda \frac{a_1^{(j)}}{|a_1^{(j)}|}} \right|^{|a_1^{(j)}|} \xrightarrow{j \to \infty} 0.$$

We define $S_j := T_{-a_1^{(j)}}$, $j \in \mathbb{N}$. Obviously $T_{a_1^{(j)}} \circ S_j = I_Y$. For any function $g(z) = e^{\lambda z}$ of Y and for each z of K we have

$$|(S_j g)(z)| \leqslant e^{|\lambda|R} |e^{-\lambda a_1^{(j)}}| = e^{|\lambda|R} \left| e^{\lambda \frac{a_1^{(j)}}{|a_1^{(j)}|}} \right|^{-|a_1^{(j)}|} \xrightarrow{j \to \infty} 0.$$

For coordinates $i \neq 1$, we have that either $\{|a_i^{(j)}|\}_{j \geqslant 1}$ is bounded, which easily implies that $\{T_{a_i^{(j)}}\}_{j \geqslant 1}$ satisfies the TUC, or $\sup_j |a_i^{(j)}| = \infty$, which yields that $\{T_{a_i^{(j)}}\}_{j \geqslant 1}$ satisfies the UC. Summarizing, we get a subsequence (m_k) of positive integers such that $(T_{a_1^{(m_k)}})$ satisfies the UC, and for each $i \neq 1$, $(T_{a_i^{(m_k)}})$ either satisfies the TUC or the UC. The conclusion follows from Theorem 1.8.

2.2. *Tensor products of backward shifts*

Weighted backward shifts constitute an important class of operators which is the "favorite testing ground" for hypercyclicity (see [26]). The derivative operator, defined on spaces of C^∞-functions in which the polynomials are dense, can be represented as a weighted backward shift (see [19]). This motivates us to characterize hypercyclicity of tensor products of weighted backward shifts on spaces of p-summable sequences ($1 \leqslant p < \infty$).

For $1 \leqslant p < \infty$, $B : l_p \to l_p$ denotes the backward shift operator defined by $B(x_0, x_1, \ldots) := (x_1, x_2, \ldots)$. Given a sequence of numbers $\{v_i\}_{i=1}^\infty$, the associated weighted backward shift is the linear map $B_v(x_0, x_1, \ldots) := (v_1 x_1, v_2 x_2, \ldots)$. It is easy to see that a weighted backward shift is well defined and continuous if and only if $\{v_i\}_{i=1}^\infty \in l_\infty$ and we will assume this from now on. Salas [26] showed that B_v is hypercyclic on l_p if and only if $\sup_{n \in \mathbb{N}} \prod_{j=1}^n |v_j| = \infty$ ($1 \leqslant p < \infty$ or $p = 0$). Accordingly, the following result should not be too surprising.

Proposition 2.1. *Let $1 \leqslant p, q < \infty$ and let $B_v : l_p \to l_p$, $B_w : l_q \to l_q$ be two weighted backward shifts. Then*

$$B_v \widetilde{\otimes} B_w : l_p \widetilde{\otimes}_a l_q \to l_p \widetilde{\otimes}_a l_q$$

is hypercyclic for any (some) tensor norm a if and only if

$$\sup_{n \in \mathbb{N}} \prod_{j=1}^n |v_j w_j| = \infty.$$

Proof. First observe that for $(x_0, x_1, \dots) \in l_p$ and $(y_0, y_1, \dots) \in l_q$ we have $|x_0 y_0| \leqslant ||x||_p ||y||_q$. Thus the map

$$f : l_p \times l_q \to \mathbb{K} : f(x, y) := x_0 y_0$$

is a continuous bilinear map which induces a continuous linear map $f \in (l_p \otimes_a l_q)'$ that we can extend to $f \in (l_p \widetilde{\otimes}_a l_q)'$. The sequence $\{B^n \widetilde{\otimes} B^n\}_{n=1}^{\infty}$ is equicontinuous on $L(l_p \widetilde{\otimes}_a l_q, l_p \widetilde{\otimes}_a l_q)$ (see [14, 35.2]). On the other hand, given $n \in \mathbb{N}$ and $x \otimes y$ with $x \in l_p$ and $y \in l_q$,

$$\begin{aligned} f \circ (B_v^n \widetilde{\otimes} B_w^n)(x \otimes y) &= f(B_v^n x \otimes B_w^n y) \\ &= f((v_1 \cdots v_n x_n, \dots) \otimes (w_1 \cdots w_n y_n, \dots)) = \left(\prod_{j=1}^{n} v_j w_j \right) x_n y_n, \end{aligned}$$

and

$$\left(\prod_{j=1}^{n} v_j w_j \right) f \circ (B^n \widetilde{\otimes} B^n)(x \otimes y) = \left(\prod_{j=1}^{n} v_j w_j \right) x_n y_n.$$

Therefore by linearity and continuity,

$$f \circ (B_v^n \widetilde{\otimes} B_w^n) = \left(\prod_{j=1}^{n} v_j w_j \right) f \circ (B^n \widetilde{\otimes} B^n) \tag{1}$$

on $l_p \widetilde{\otimes}_a l_q$ for each $n \in \mathbb{N}$.

If $\sup_{n \in \mathbb{N}} \prod_{j=1}^{n} |v_j w_j| < \infty$, by (1) and the equicontinuity of $\{B^n \widetilde{\otimes} B^n\}_{n=1}^{\infty}$ we then have that $\{f \circ (B_v^n \widetilde{\otimes} B_w^n)\}_{n \in \mathbb{N}}$ is equicontinuous on $(l_p \widetilde{\otimes}_a l_q)'$. Thus for any $z \in l_p \widetilde{\otimes}_a l_q$ the set $\{f \circ (B_v^n \widetilde{\otimes} B_w^n)(z)\}_{n \in \mathbb{N}}$ is bounded on $\mathbb{K}$, which implies that z cannot be hypercyclic for $B_v \widetilde{\otimes} B_w$.

Conversely, if $\sup_{n \in \mathbb{N}} \prod_{j=1}^{n} |v_j w_j| = \infty$, we set

$$X = Y := \left\{ \sum_{i,j=1}^{n} \lambda_{i,j} e_i \otimes e_j : \lambda_{i,j} \in \mathbb{K},\ n \in \mathbb{N} \right\},$$

and define the forward shifts $S_v(e_{i-1}) := (1/v_i) e_i,\ S_w(e_{i-1}) := (1/w_i) e_i,\ i \in \mathbb{N}$. Note that $\sup_{n \in \mathbb{N}} \prod_{j=1}^{n} |v_j w_j| = \infty$ implies $v_i \neq 0 \neq w_i$ for all $i \in \mathbb{N}$. Now the map $S := S_v \otimes S_w : Y \to Y$ is well defined. To show that $B_v \otimes B_w$ satisfies the HC, we observe:

(i) $B_v^m \otimes B_w^m (e_i \otimes e_j)$ is eventually 0, therefore $B_v^m \otimes B_w^m x \xrightarrow{m \to \infty} 0$ for each $x \in X$.

(ii) If we consider the constants

$$C_1 := 1 + \sup_{i \in \mathbb{N}} |v_i|, \quad C_2 := 1 + \sup_{i \in \mathbb{N}} |w_i|,$$

then our hypothesis implies

$$\exists m_1(>1) : \prod_{i=1}^{m_1} |v_i w_i| > C_1 C_2$$
$$\vdots$$
$$\exists m_k(>m_{k-1}+k) : \prod_{i=1}^{m_k} |v_i w_i| > k C_1^k C_2^k$$
$$\vdots$$

Let $n_k := m_k - k,\ k \in \mathbb{N}$. Given $i,\ j \in \mathbb{N}$, we define

$$D_{i,j} := \prod_{\substack{r \leqslant i \\ s \leqslant j}} |v_r w_s|,$$

and we take $k > \max\{i,j\}$. Then

$$||S^{n_k}(e_i \otimes e_j)|| = \frac{1}{\prod_{r=1}^{n_k} |v_{i+r} w_{j+r}|} ||e_{i+n_k} \otimes e_{j+n_k}||$$
$$= D_{i,j} \frac{1}{\prod_{r=1}^{i+n_k} |v_r| \prod_{s=1}^{j+n_k} |w_s|} < D_{i,j} \frac{C_1^k C_2^k}{\prod_{r=1}^{m_k} |v_r| \prod_{s=1}^{m_k} |w_s|} < \frac{D_{i,j}}{k} \overset{k \to \infty}{\longrightarrow} 0.$$

Hence $S^{n_k} y \overset{k \to \infty}{\longrightarrow} 0$ for each $y \in X$.

(iii) The definition of S gives $(B_v \widetilde{\otimes} B_w) \circ S = I_Y$.

This means that $B_v \widetilde{\otimes} B_w$ satisfies the HC with respect to the sequence (n_k) that we selected, and therefore it is hypercyclic. □

Examples 2.2. (1) The tensor product of two hypercyclic operators is not necessarily hypercyclic. The pair of weights $v = (v_j) := (2, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \ldots)$ and $w = (w_j) := (\frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, satisfy $\sup_{n \in \mathbb{N}} \prod_{j=1}^n v_j = \infty$ and $\sup_{n \in \mathbb{N}} \prod_{j=1}^n w_j = \infty$, so B_v and B_w are hypercyclic by [26, Theorem 2.8]. Clearly $\prod_{j=1}^n v_j w_j = 1$ for all $n \in \mathbb{N}$ and by Proposition 2.1 we obtain that $B_v \widetilde{\otimes} B_w$ is not hypercyclic.

(2) The condition for hypercyclicity of tensor products of weighted shifts clearly implies that at least one of the factors satisfies the HC. This is probably all we can say about the factors. For example, if we define $v := (1, 2, 2, 1/2^3, 1, 1, 1, 2, 2, 2, 2, 2, 2, 1/2^7, 1, 1, 1, 1, 1, 1, 1, \ldots)$, and the other weight $w := (1, 1/2, 1, 2, 2, 2, 2, 1/2^5, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots)$, then we have that $B_v \widetilde{\otimes} B_w$ is hypercyclic, and both B_v and B_w satisfy the HC. However, a careful analysis shows that there is no (n_k) such that one of the weighted shifts satisfies the HC, and the other one satisfies the THC, both with respect to (n_k).

2.3. Multiplication operators on $L(E)$

Chan [12] studied the hypercyclicity of the left multiplication operator $L_T(S) := T \circ S$ on the algebra $L(H)$ (H Hilbert) endowed with the strong operator topology. The proof of his result was direct assuming the HC on T. The surprising fact is that no Baire argument is available in $L(H)$ endowed with this topology as it is far from being a Fréchet space. This result was generalized to the algebra of bounded operators on arbitrary separable Banach spaces in [13]. We will characterize universality of left multiplication operators by using tensor product techniques, which differs from Chan's approach. In [10] we extend our study to right multiplication operators, ideals of $L(E)$, and we also treat non-Banach spaces E.

We recall the natural inclusion of $E^* \otimes E$ in $L(E)$: $f(e) := \sum \langle e_i^*, e \rangle e_i$ if $f = \sum e_i^* \otimes e_i \in E^* \otimes E$. This inclusion is dense if we consider the strong operator topology (SOT) on $L(E)$, i.e. the topology of pointwise convergence. Indeed, if $S \in L(E)$ and $x_1, \dots, x_n \in E$, we fix a projection $P : E \to \text{span}\{x_1, \dots, x_n\}$ and then $S \circ P \in E^* \otimes E$, $S \circ Px_i = Sx_i$, $i = 1, \dots, n$. $E^* \otimes E$, as a subspace of $L(E)$ with the operator norm, inherits the injective topology $E^* \otimes_\varepsilon E$ associated with the tensor norm ε, which is the coarsest tensor norm. Moreover, the restriction of L_T to $E^* \otimes E$ coincides with $I \otimes T$.

Theorem 2.3. *Let $\{T_n : E \to E,\ n \in \mathbb{N}\}$ be a commuting sequence of operators defined on a separable Banach space E. Let us consider the left multiplication operators*

$$L_{T_n} : L(E) \to L(E) : S \mapsto T_n \circ S, \quad n \in \mathbb{N},$$

on the algebra $L(E)$ endowed with the SOT. The following are equivalent:

(i) *$\{T_n\}_{n=1}^{\infty}$ satisfies the UC,*
(ii) *$\{L_{T_n}\}_{n=1}^{\infty}$ is universal.*

In particular, if $T : E \to E$ is an operator, then the following are equivalent:

(i) *T satisfies the HC (respectively, is chaotic),*
(ii) *L_T is hypercyclic (chaotic).*

Proof. (i) → (ii) We first observe that E and $(E^*, \sigma(E^*, E))$ are separable. Indeed, we have that the closed unit ball B of E^*, being $\sigma(E^*, E)$-compact and $\sigma(E^*, E)$-metrizable, is $\sigma(E^*, E)$-separable. We pick a countable weak-* dense subset $X \subset E^*$ and define Y as the $||.||^*$-closure of X. Then $F := (Y, ||.||^*) \widetilde{\otimes}_\varepsilon E$ is a separable Banach space. The extension $F \to \overline{E^* \otimes E}^{L(E)}$ of the continuous inclusion gives us an operator $\phi : F \to L(E)$ with dense range since $\text{span}\{x \otimes e,\ x \in X,\ e \in E\}$ is dense in $L(E)$. By Theorem 1.8 $\{I \widetilde{\otimes} T_n\}_{n=1}^{\infty}$ is universal on F. The implication follows by applying Lemma 1.3 to the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{I\tilde{\otimes}T_n} & F \\ \phi\big\downarrow & & \phi\big\downarrow \\ L(E) & \xrightarrow{L_{T_n}} & L(E) \end{array} . \qquad (2)$$

(ii)$\to$(i) If $\{L_{T_n}\}_{n=1}^{\infty}$ is universal on $L(E)$ we then pick $x, y \in E$ such that $\{x, y\}$ is linearly independent and consider the following commutative diagram:

$$\begin{array}{ccc} L(E) & \xrightarrow{L_{T_n}} & L(E) \\ \Psi\big\downarrow & & \Psi\big\downarrow \\ E\oplus E & \xrightarrow{T_n\oplus T_n} & E\oplus E \end{array}$$

where $\Psi(R) := (Rx, Ry),\ R \in L(E)$, and Ψ is surjective. By a recent result of Bernal and Grosse–Erdmann [5, Theorem 3.3] which improves a previous result of Bès and the second author [6, Remark 2.6 (3)], we conclude that $\{T_n\}_{n=1}^{\infty}$ satisfies the UC since $\{T_n \oplus T_n\}_{n=1}^{\infty}$ is universal.

The chaotic case follows from the simple observation that T is chaotic whenever $T \oplus T$ does. $\square$

2.4. Composition operators in infinite holomorphy

We finally give examples of hypercyclic and chaotic composition operators in infinite holomorphy as a consequence of our tensor techniques. The first examples of hypercyclic operators defined on a Fréchet space of entire functions on an infinite dimensional Banach space were given by Aron and Bès [2]. More precisely they showed that translation operators are hypercyclic on a certain Fréchet algebra of entire functions on a Banach space. Here we will treat a different kind of composition operators: Namely, composition with a linear operator.

We recall that, given a complex Banach space E, the space of entire functions on E is

$$\mathscr{H}(E) := \{f : E \to \mathbb{C} : f \text{ is continuous and } f_{|_F} \text{ is}$$
$$\text{holomorphic for each } F \hookrightarrow E \text{ finite dimensional}\}.$$

We will consider on $\mathscr{H}(E)$ the topology τ_o of uniform convergence on compact sets on E. The space of bounded entire functions on E is

$$\mathscr{H}_b(E) := \{f \in \mathscr{H}(E) : f(B) \text{ bounded for each } B \subset E \text{ bounded}\}.$$

This space is usually endowed with the topology τ_b of uniform convergence on the bounded sets of E. If $f \in \mathscr{H}(E)$ (respectively $f \in \mathscr{H}_b(E)$) then f admits the expression as a series

$$f(z) = \sum_{n=0}^{\infty} \frac{\widehat{d^n f(0)}}{n!}(z), \quad z \in E,$$

where the series converges with respect τ_o (τ_b) and

$$\frac{\widehat{d^n} f(0)}{n!}(z) = \frac{1}{2\pi i}\int_{|\lambda|=1} \frac{f(\lambda z)}{\lambda^{n+1}}\, d\lambda.$$

Moreover, the map

$$P_n(f) := \widehat{d^n} f(0) : E \to \mathbb{C} : z \mapsto \widehat{d^n} f(0)(z)$$

is a continuous n-homogeneous polynomial.

The space of symmetric n-tensors on a Banach space F is defined by

$$\bigotimes_{n,s} F = \left\{ \sum_{i\in M} z_i \otimes \cdots \otimes z_i : M \text{ finite}, \ z_i \in F \right\} \hookrightarrow F \otimes \cdots \otimes F.$$

We denote by $\otimes_{n,s,\varepsilon} F$ the space of symmetric n-tensors endowed with the injective tensor norm ε. Then $\otimes_{n,s,\varepsilon} E^*$ is a topological subspace of the space $(\mathscr{P}(^nE), \tau_b)$ of n-homogeneous polynomials on E with the topology of uniform convergence on bounded sets of E. Indeed, one just has to identify

$$\begin{array}{ccccc} \otimes_{n,s} E^* & \to & \mathscr{P}(^nE) & & \\ z^* \otimes \cdots \otimes z^* & \mapsto & P : E & \to & \mathbb{C} \\ & & z & \mapsto & \langle z, z^* \rangle^n \end{array}$$

Via this identification, if we define

$$G := \left\{ f \in \mathscr{H}_b(E) : P_n(f) \in \bigotimes_{n,s} E^*, \ \forall n \in \mathbb{N} \right\} \hookrightarrow (\mathscr{H}_b(E), \tau_b),$$

then the Fréchet space $\mathscr{H}_{bc}(E) := \bar{G}^{\tau_b}$ is the algebra of entire functions of compact type, i.e. the Fréchet algebra generated by the elements of E^*. We notice that $\mathscr{H}_{bc}(E)$ is precisely the algebra on which Aron and Bès studied hypercyclicity of translation operators. Our purpose is to study the composition operator $R_T(f) := f \circ T$, for T linear. Observe that in this case $f(T(0)) = f(0)$. Therefore to get hypercyclicity we have to consider functions which fix zero. This forces us to define $\mathscr{H}_0(E) := \{f \in \mathscr{H}(E) : f(0) = 0\}$ and $\mathscr{H}_{bc0}(E) := \{f \in \mathscr{H}_{bc}(E) : f(0) = 0\}$.

Theorem 2.4. *Let E be a Banach space with separable dual E^* and $T : E \to E$ an operator such that its transpose $T^* : E^* \to E^*$ satisfies the Hypercyclicity Criterion* (*is chaotic*). *Then the composition operator*

$$R_T : (\mathscr{H}_{bc0}(E), \tau_b) \to (\mathscr{H}_{bc0}(E), \tau_b), \quad f \mapsto f \circ T,$$

is hypercyclic (*chaotic*). *Conversely, if R_T is hypercyclic* (*chaotic*) *then we have that $T^* : E^* \to E^*$ is hypercyclic* (*chaotic*). *If E has the approximation property and T^* satisfies the Hypercyclicity Criterion* (*is chaotic*), *then R_T is hypercyclic* (*chaotic*) *on* $(\mathscr{H}_0(E), \tau_o)$.

Proof. We recall that

$$G := \left\{ f \in \mathscr{H}_b(E) : f(0) = 0,\ P_n(f) \in \bigotimes_{n,s} E^*,\ \forall n \in \mathbb{N} \right\}$$

is a dense subspace of $\mathscr{H}_{bc0}(E)$. If $f \in G$, then $P_n(R_T(f)) = P_n(f) \circ T = (T^* \otimes \cdots \otimes T^*)(P_n(f)) \in \otimes_{n,s} E^*$ for all $n \in \mathbb{N}$. This shows $R_T(G) \subset G$ and therefore R_T is well-defined.

The extension $\widetilde{\otimes}_{n,s,\varepsilon} T^* =: T_n^*$ of $T^* \otimes \cdots \otimes T^*$ to $\widetilde{\otimes}_{n,s,\varepsilon} E^* = \overline{\otimes_{n,s} E^*}^{\tau_b}$ satisfies the HC (is chaotic) with respect to a sequence (m_k) (independent on n), by a similar argument to the one of Theorem 1.8. Consequently there are $X_n, Y_n \subset \widetilde{\otimes}_{n,s,\varepsilon} E^*$ dense and $S_{n,m_k} : Y_n \to \widetilde{\otimes}_{n,s,\varepsilon} E^*$, $k \in \mathbb{N}$, such that T_n^*, X_n, Y_n, and $\{S_{n,m_k}\}_{k=1}^{\infty}$ satisfy (i)–(iii) of Definition 1.1, $n \in \mathbb{N}$.

Define now $X := \bigcup_{n \in \mathbb{N}} (\oplus_{k=1}^n X_k)$, $Y := \bigcup_{n \in \mathbb{N}} (\oplus_{k=1}^n Y_k)$ (dense subspaces of $\mathscr{H}_{bc0}(E)$), $S_{m_k} : Y \to \mathscr{H}_{bc0}(E)$, $S_{m_k} := \oplus_{n \in \mathbb{N}} S_{n,m_k}$, $k \in \mathbb{N}$. It easily follows that R_T, X, Y, and $\{S_{m_k}\}_{k=1}^{\infty}$ satisfy the conditions of the HC, therefore R_T is hypercyclic (chaotic) on $\mathscr{H}_{bc0}(E)$.

If R_T is hypercyclic, we consider the commutative diagram

$$\begin{array}{ccc}
\mathscr{H}_{bc0}(E) & \xrightarrow{R_T} & \mathscr{H}_{bc0}(E) \\
{\scriptstyle P_1}\big\downarrow & & {\scriptstyle P_1}\big\downarrow \\
E^* & \xrightarrow{T^*} & E^*
\end{array}$$

where P_1 is the surjective map which associates the corresponding 1-homogeneous polynomial to each entire function. Our Lemma 1.3 does the job.

If E has the approximation property and T^* satisfies the HC, once again the comparison principle yields the conclusion since $\mathscr{H}_{bc0}(E) \hookrightarrow \mathscr{H}_0(E)$ is dense (see, e.g., [16, Example 2.79]). $\square$

We observe that our result remains valid for (DF)-spaces E with separable strong dual E'_b. This is a consequence of the fact that, in this case, $(\mathscr{H}_b(E), \tau_b)$ is also a Fréchet space by a result of Galindo et al. [17].

Acknowledgments

This paper was completed while Alfredo Peris was visiting the Department of Mathematics at Michigan State University. He acknowledges the hospitality. We thank Andreas Defant and Klaus Floret for some interesting discussions about the topic of this paper that we had four years ago. We are specially indebted to José Bonet for providing us ideas and suggestions that improved our note, particularly concerning the results about the algebra $L(E)$. We also thank the referees who contributed to improve the presentation of the paper.

References

[1] Y. Abe, P. Zappa, Universal functions on complex general linear groups, J. Approx. Theory 100 (1999) 221–232.
[2] R. Aron, J. Bés, Hypercyclic differentiation operators, Function Spaces (Edwardsville, IL, 1998), Contemp. Math. 232 (1999) 39–46.
[3] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (4) (1992) 332–334.
[4] L. Bernal-González, Hypercyclic sequences of differential and antidifferential operators, J. Approx. Theory 96 (2) (1999) 323–337.
[5] L. Bernal-González, K.G. Grosse-Erdmann, The hypercyclicity criterion for sequences of operators, Studia Math., to appear.
[6] J.P. Bès, A. Peris, Hereditarily hypercyclic operators, J. Funct. Anal. 167 (1) (1999) 94–112.
[7] G.D. Birkhoff, Démonstration d'un théoréme elémentaire sur les fonctions entiéres, C. R. Acad. Sci. Paris 189 (1929) 473–475.
[8] J. Bonet, Hypercyclic and chaotic convolution operators, J. London Math. Soc. 62 (2) (2000) 253–262.
[9] J. Bonet, F. Martínez-Giménez, A. Peris, A Banach space which admits no chaotic operator, Bull. London Math. Soc. 33 (2001) 196–198.
[10] J. Bonet, F. Martínez-Giménez, A. Peris, Universal and chaotic multipliers on operator ideals, preprint, 2001.
[11] J. Bonet, A. Peris, Hypercyclic operators on non-normable Fréchet spaces, J. Funct. Anal. 159 (2) (1998) 587–595.
[12] K.C. Chan, Hypercyclicity of the operator algebra for a separable Hilbert space, J. Oper. Theory 42 (1999) 231–244.
[13] K.C. Chan, R.D. Taylor, Hypercyclic subspaces of a Banach space, Integral Equations Operator Theory 41 (2001) 381–388.
[14] A. Defant, K. Floret, Tensor Norms and Operator Ideals, North-Holland, Amsterdam, 1993.
[15] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd Edition, Addison-Wesley, Reading, MA, 1989.
[16] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer, London, 1999.
[17] P. Galindo, D. García, M. Maestre, Holomorphic mappings of bounded type on (DF)-spaces, in: K.D. Bierstedt, J. Bonet, J. Horváth, M. Maestre (Eds.), Progress in Functional Analysis, North-Holland Mathematics Studies, Vol. 170, North-Holland, Amsterdam, New York, 1992, pp. 135–148.
[18] R.M. Gethner, J.H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. 100 (2) (1987) 281–288.
[19] G. Godefroy, J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (2) (1991) 229–269.
[20] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (3) (1999) 345–381.
[21] D.A. Herrero, Hypercyclic operators and chaos, J. Operator Theory 28 (1) (1992) 93–103.
[22] C. Kitai, Invariant closed sets for linear operators, Ph.D. Thesis, University of Toronto, 1982.
[23] F. Martínez-Giménez, Universalidad, Hiperciclicidad, y Caos en Espacios de Fréchet, Ph.D. Thesis, Universitat Politècnica de València, 2000.
[24] F. Martínez-Giménez, A. Peris, Chaos for backward shift operators, Int. J. Bifurcation Chaos 12 (2002) 1703–1715.
[25] N. Nikolski, Remarks concerning completeness of translates in function spaces, J. Approx. Theory 98 (1999) 303–315.
[26] H.N. Salas, Hypercyclic weighted shifts, Trans. Amer. Math. Soc. 347 (3) (1995) 993–1004.
[27] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer, New York, 1993.